

THE BETTI NUMBERS FOR A FAMILY OF SOLVABLE LIE ALGEBRAS

THANH MINH DUONG

ABSTRACT. We give a characterization of symplectic quadratic Lie algebras that their Lie algebra of inner derivations has an invertible derivation. A family of symplectic quadratic Lie algebras is introduced to illustrate this situation. Finally, we calculate explicitly the Betti numbers of a family of solvable Lie algebras in two ways: using the cohomology of quadratic Lie algebras and applying a Pouseele's result on extensions of the one-dimensional Lie algebra by Heisenberg Lie algebras.

0. INTRODUCTION

Let \mathfrak{g} be a complex Lie algebra endowed with a non-degenerate invariant symmetric bilinear form B , $\{X_1, \dots, X_n\}$ be a basis of \mathfrak{g} and $\{\omega_1, \dots, \omega_n\}$ be its dual basis. Denote by $\{Y_1, \dots, Y_n\}$ the basis of \mathfrak{g} defined by $B(Y_i, \cdot) = \omega_i$, $1 \leq i \leq n$. Pinczon and Ushirobira discovered in [5] that the differential ∂ on $\wedge(\mathfrak{g}^*)$, the space of antisymmetric forms on \mathfrak{g} , is given by $\partial := -\{I, \cdot\}$ where I is defined by:

$$I(X, Y, Z) = B([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g}$$

and $\{, \}$ is the super Poisson bracket on $\wedge(\mathfrak{g}^*)$ defined by

$$\{\Omega, \Omega'\} = (-1)^{k+1} \sum_{i,j} B(Y_i, Y_j) \iota_{X_i}(\Omega) \wedge \iota_{X_j}(\Omega'), \quad \forall \Omega \in \wedge^k(\mathfrak{g}^*), \Omega' \in \wedge^l(\mathfrak{g}^*).$$

In Section 1, by using this, we detail a result of Medina and Revoy in [4] that there is an isomorphism between the second cohomology group $H^2(\mathfrak{g}, \mathbb{C})$ and $\text{Der}_a(\mathfrak{g})/\text{ad}(\mathfrak{g})$ where $\text{Der}_a(\mathfrak{g})$ is the vector space of skew-symmetric derivations of \mathfrak{g} and $\text{ad}(\mathfrak{g})$ is its subspace of inner ones.

Involving in the well-known theorem by Jacobson on the invertibility of Lie algebra derivations that a Lie algebra over a field of characteristic zero is nilpotent if it admits an invertible derivation, we are interested in Lie algebras having an invertible derivation. We prove that the Lie algebra $\text{ad}(\mathfrak{g})$ of a symplectic quadratic Lie algebra has that property. In particular, we have the following (Proposition 1.5).

THEOREM 1. *Let $(\mathfrak{g}, B, \omega)$ be a symplectic quadratic Lie algebra. Consider the mapping $\mathcal{D} : \text{ad}(\mathfrak{g}) \rightarrow \text{ad}(\mathfrak{g})$ defined by $\mathcal{D}(\text{ad}(X)) = \text{ad}(\phi^{-1}(\iota_X(\omega)))$ with $\phi : \mathfrak{g} \rightarrow \mathfrak{g}^*$, $\phi(X) = B(X, \cdot)$, then \mathcal{D} is an invertible derivation of $\text{ad}(\mathfrak{g})$.*

The reader is referred to [2] for further information about symplectic quadratic Lie algebras. A family of such algebras is given to illustrate this situation.

In Section 2, motivated by Corollary 4.4 in [4], we give the Betti numbers for a family of solvable quadratic Lie algebras defined as follows. For each $n \in \mathbb{N}$, let \mathfrak{g}_{2n+2} denote

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the Lie algebra with basis $\{X_0, \dots, X_n, Y_0, \dots, Y_n\}$ and non-zero Lie brackets $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$, $[X_i, Y_i] = X_0$, $1 \leq i \leq n$. Denote by $B^k(\mathfrak{g}_{2n+2}) = B^k(\mathfrak{g}_{2n+2}, \mathbb{C})$, $Z^k(\mathfrak{g}_{2n+2}) = Z^k(\mathfrak{g}_{2n+2}, \mathbb{C})$, $H^k(\mathfrak{g}_{2n+2}) = H^k(\mathfrak{g}_{2n+2}, \mathbb{C})$ and $b_k = b_k(\mathfrak{g}_{2n+2}, \mathbb{C})$. By computing on super Poisson brackets, our second result is the following.

THEOREM 2. *The k^{th} Betti numbers of \mathfrak{g}_{2n+2} are given as follows:*

(1) *If k is even then one has*

$$b_k = \left| \binom{n}{\frac{k}{2}} \binom{n}{\frac{k}{2}} - \binom{n}{\frac{k-2}{2}} \binom{n}{\frac{k-2}{2}} \right|.$$

(2) *If k is odd then one has*

• *if $k < n+1$ then*

$$b_k = \binom{n}{\frac{k-1}{2}} \binom{n}{\frac{k-1}{2}} - \binom{n}{\frac{k-3}{2}} \binom{n}{\frac{k-3}{2}},$$

• *if $k = n+1$ then*

$$b_{n+1} = 2 \binom{n}{\frac{n}{2}} \binom{n}{\frac{n}{2}} - 2 \binom{n}{\frac{n+2}{2}} \binom{n}{\frac{n+2}{2}},$$

• *if $k > n+1$ then*

$$b_k = \binom{n}{\frac{k-1}{2}} \binom{n}{\frac{k-1}{2}} - \binom{n}{\frac{k+1}{2}} \binom{n}{\frac{k+1}{2}}.$$

Our method is direct and different from the Pouseele's method given in [6] that we shall recall in Appendix 1. In the Pouseele's method, the Betti numbers of \mathfrak{g}_{2n+2} follow the Betti numbers of the $2n+1$ -dimensional Lie algebra \mathfrak{f} defined by $[x, x_i] = x_i$ and $[y, y_i] = -y_i$ for all $1 \leq i \leq n$.

Other results of Betti numbers for some families of nilpotent Lie algebras, we refer the reader to [1], [6] or [7].

1. A CHARACTERIZATION OF SYMPLECTIC QUADRATIC LIE ALGEBRAS

Let \mathfrak{g} be a complex Lie algebra endowed with a non-degenerate invariant symmetric bilinear form B . In this case, we call the pair (\mathfrak{g}, B) a *quadratic* Lie algebra. Denote by $\text{Der}_a(\mathfrak{g})$ the vector space of skew-symmetric derivations of \mathfrak{g} , that is the vector space of derivations D satisfying $B(D(X), Y) = -B(X, D(Y))$ for all $X, Y \in \mathfrak{g}$, then $\text{Der}_a(\mathfrak{g})$ is a Lie subalgebra of $\text{Der}(\mathfrak{g})$.

Proposition 1.1. *There exists a Lie algebra isomorphism T between $\text{Der}_a(\mathfrak{g})$ and the space $\{\Omega \in \wedge^2(\mathfrak{g}^*) \mid \{I, \Omega\} = 0\}$. This isomorphism induces an isomorphism from $\text{ad}(\mathfrak{g})$ onto $\iota_{\mathfrak{g}}(I) = \{\iota_X(I) \in \wedge^2(\mathfrak{g}^*) \mid X \in \mathfrak{g}\}$.*

Proof. Let $D \in \text{Der}_a(\mathfrak{g})$ and set $\Omega \in \wedge^2(\mathfrak{g}^*)$ by $\Omega(X, Y) = B(D(X), Y)$ for all $X, Y \in \mathfrak{g}$. Then D is a derivation of \mathfrak{g} if and only if

$$\Omega([X, Y], Z) + \Omega([Y, Z], X) + \Omega([Z, X], Y) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. It means $\{I, \Omega\} = 0$. Define the map T from $\text{Der}_a(\mathfrak{g})$ onto $\{\Omega \in \wedge^2(\mathfrak{g}^*) \mid \{I, \Omega\} = 0\}$ by $T(D) = \Omega$ then T is a one-to-one correspondence.

Now we shall show that $T([D, D']) = \{T(D), T(D')\}$ for all $D, D' \in \text{Der}_a(\mathfrak{g})$. Indeed, set $\Omega = T(D)$, $\Omega' = T(D')$ and fix an orthonormal basis $\{X_j\}_{j=1}^n$ of \mathfrak{g} . One has

$$\begin{aligned} \{\Omega, \Omega'\}(X, Y) &= - \left(\sum_{j=1}^n \iota_{X_j}(\Omega) \wedge \iota_{X_j}(\Omega') \right) (X, Y) \\ &= - \sum_{j=1}^n (\Omega(X_j, X) \Omega'(X_j, Y) - \Omega(X_j, Y) \Omega'(X_j, X)) \\ &= - \sum_{j=1}^n B(B(D(X_j), X) D'(X_j) - B(D'(X_j), X) D(X_j), Y) \\ &= - \sum_{j=1}^n B(D'(D(X)) - D(D'(X)), Y) = -B([D', D](X), Y). \end{aligned}$$

That means $T([D, D']) = \{T(D), T(D')\}$ and then T is a Lie algebra isomorphism.

If $D = \text{ad}(X_0)$ then $T(D)(X, Y) = B([X_0, X], Y) = I(X_0, Y, Z) = \iota_{X_0}(I)(X, Y)$. Therefore, $T(D) = \iota_{X_0}(I)$. \square

Corollary 1.2. $\{\iota_X(I), \iota_Y(I)\} = \iota_{[X, Y]}(I)$.

Corollary 1.3. [4]

The cohomology group $H^2(\mathfrak{g}, \mathbb{C}) \simeq \text{Der}_a(\mathfrak{g}, B)/\text{ad}(\mathfrak{g})$.

Definition 1.4. A non-degenerate skew-symmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is called a *symplectic structure* on \mathfrak{g} if it satisfies

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

A symplectic structure ω on a quadratic Lie algebra (\mathfrak{g}, B) is corresponding to a skew-symmetric invertible derivation D defined by $\omega(X, Y) = B(D(X), Y)$, for all $X, Y \in \mathfrak{g}$. As above, a symplectic structure is exactly a non-degenerate 2-form ω satisfying $\{I, \omega\} = 0$. If \mathfrak{g} has a such ω then we call $(\mathfrak{g}, B, \omega)$ a *symplectic quadratic Lie algebra*.

For symplectic quadratic Lie algebras, the reader can refer to [2] for more details. Here we give a following property.

Proposition 1.5. *Let $(\mathfrak{g}, B, \omega)$ be a symplectic quadratic Lie algebra. Consider the mapping $\mathcal{D} : \text{ad}(\mathfrak{g}) \rightarrow \text{ad}(\mathfrak{g})$ defined by $\mathcal{D}(\text{ad}(X)) = \text{ad}(\phi^{-1}(\iota_X(\omega)))$ with $\phi : \mathfrak{g} \rightarrow \mathfrak{g}^*$, $\phi(X) = B(X, \cdot)$, then \mathcal{D} is an invertible derivation of $\text{ad}(\mathfrak{g})$.*

Proof. As above we have $\{I, \omega\} = 0$ and then $\iota_X(\{I, \omega\}) = 0$ for all $X \in \mathfrak{g}$. It implies $\{\iota_X(I), \omega\} = \{I, \iota_X(\omega)\}$ for all $X \in \mathfrak{g}$. Note that if X is nonzero, since ω is non-degenerate then $\iota_X(\omega)$ is non trivial. Set $Y = \phi^{-1}(\iota_X(\omega))$ then $\{I, \iota_X(\omega)\} = \iota_Y(I)$ and therefore this defines an inner derivation. Let D be the derivation corresponding to ω then one has $[\text{ad}(X), D] = \text{ad}(Y)$.

Let $\text{ad}(X) \in \text{ad}(\mathfrak{g})$. Set $\alpha = \phi(X)$. Since ω is non-degenerate then there exists an element $Y \in \mathfrak{g}$ such that $\alpha = \iota_Y(\omega)$. In this case, $\mathcal{D}(\text{ad}(Y)) = \text{ad}(X)$. That means \mathcal{D} onto and therefore it is bijective. \square

Next, we give a family of symplectic quadratic Lie algebras that has been defined in [3] as follows.

Example 1.6. Let $p \in \mathbb{N} \setminus \{0\}$. We denote the *Jordan block of size p* by $J_1 := (0)$ and for $p \geq 2$,

$$J_p := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

For $p \geq 2$, we consider $\mathfrak{q} = \mathbb{C}^{2p}$ with a basis $\{X_i, Y_i\}$, $1 \leq i \leq p$, and equipped with a bilinear form B satisfying $B(X_i, X_j) = B(Y_i, Y_j) = 0$ and $B(X_i, Y_j) = \delta_{ij}$. Let $C : \mathfrak{q} \rightarrow \mathfrak{q}$ with matrix

$$C = \begin{pmatrix} J_p & 0 \\ 0 & -{}^t J_p \end{pmatrix}$$

in the given basis. Then $C \in \mathfrak{o}(2p)$.

Let $\mathfrak{h} = \mathbb{C}^2$ and $\{X_0, Y_0\}$ be a basis of \mathfrak{h} . Define on the vector space $\mathfrak{j}_{2p} = \mathfrak{q} \oplus \mathfrak{h}$ the Lie bracket $[Y_0, X] = C(X)$, $[X, Y] = B(C(X), Y)X_0$ and the bilinear form $\bar{B}(X_0, Y_0) = 1$, $\bar{B}(X_0, X_0) = \bar{B}(Y_0, Y_0) = \bar{B}(X_0, X) = \bar{B}(Y_0, X) = 0$ and $\bar{B}(X, Y) = B(X, Y)$ for all $X, Y \in \mathfrak{q}$. So \mathfrak{j}_{2p} is a nilpotent Lie algebra and it will be called a $2p+2$ -dimensional *nilpotent Jordan-type* Lie algebra.

Denote by $\{\alpha, \alpha_1, \dots, \alpha_p, \beta, \beta_1, \dots, \beta_p\}$ the dual basis of $\{X_0, \dots, X_p, Y_0, \dots, Y_p\}$ then $I = \beta \wedge \sum_{i=1}^{p-1} \alpha_{i+1} \wedge \beta_i$. In this case, we choose $\omega = \alpha \wedge \beta + \sum_{i=1}^p i \alpha_i \wedge \beta_i$ then $\{I, \omega\} = 0$ and therefore $(\mathfrak{j}_{2p}, B, \omega)$ is a symplectic quadratic Lie algebra. Notice that if we define $\mathcal{D}(\text{ad}(Y_0)) = -\text{ad}(Y_0)$, $\mathcal{D}(\text{ad}(X_i)) = i \text{ad}(X_i)$ and $\mathcal{D}(\text{ad}(Y_i)) = -i \text{ad}(Y_i)$ then \mathcal{D} is an invertible derivation of $\text{ad}(\mathfrak{j}_{2p})$.

2. THE BETTI NUMBERS FOR A FAMILY OF SOLVABLE QUADRATIC LIE ALGEBRAS

For each $n \in \mathbb{N}$, let \mathfrak{g}_{2n+2} denote the Lie algebra with basis $\{X_0, \dots, X_n, Y_0, \dots, Y_n\}$ and non-zero Lie brackets $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$, $[X_i, Y_i] = X_0$, $1 \leq i \leq n$. Then \mathfrak{g} is quadratic with invariant bilinear form B given by $B(X_i, Y_i) = 1$, $0 \leq i \leq n$, zero otherwise.

Let $\{\alpha, \alpha_1, \dots, \alpha_n, \beta, \beta_1, \dots, \beta_n\}$ be the dual basis of $\{X_0, \dots, X_n, Y_0, \dots, Y_n\}$ and set $V = \text{span}\{\alpha_i\}$, $W = \text{span}\{\beta_i\}$, $1 \leq i \leq n$. It is easy to check that the associated 3-form of \mathfrak{g}_{2n+2} :

$$I = \beta \wedge \sum_{i=1}^n \alpha_i \wedge \beta_i.$$

Denote by $\Omega_n := \sum_{i=1}^n \alpha_i \wedge \beta_i$ then one has

$$B^2(\mathfrak{g}_{2n+2}) = \{t_X(I) \mid X \in \mathfrak{g}_{2n+2}\} = \text{span}\{\beta \wedge \alpha_i, \beta \wedge \beta_i, \Omega_n \mid 1 \leq i \leq n\}.$$

If $n = 1$ then by we can directly calculate that $H^2(\mathfrak{g}_4) = \{0\}$. If $n > 1$, we have the non-zero super Poisson brackets:

- (i) $\{I, \alpha \wedge \alpha_i\} = \alpha_i \wedge \Omega_n - \alpha \wedge \beta \wedge \alpha_i$ and $\{I, \alpha \wedge \beta_i\} = \beta_i \wedge \Omega_n + \alpha \wedge \beta \wedge \beta_i$,
- (ii) $\{I, \alpha \wedge \beta\} = I$,
- (iii) $\{I, \alpha_i \wedge \alpha_j\} = 2\beta \wedge \alpha_i \wedge \alpha_j$ and $\{I, \beta_i \wedge \beta_j\} = -2\beta \wedge \beta_i \wedge \beta_j$.

It results that $Z^2(\mathfrak{g}_{2n+2}) = \text{span}\{\beta \wedge \alpha_i, \beta \wedge \beta_i, \alpha_i \wedge \beta_j \mid 1 \leq i, j \leq n\}$ and then the second cohomology group $H^2(\mathfrak{g}_{2n+2}) = \text{span}\{[\alpha_i \wedge \beta_j]\} / \text{span}\{[\sum_{i=1}^n \alpha_i \wedge \beta_i]\}$, where $1 \leq i, j \leq n$. So we recover the result of Medina and Revoy in [4] obtained by describing the space $\text{Der}_a(\mathfrak{g}_{2n+2})$ that $b_2 = n^2 - 1$.

To get the Betti numbers b_k for $k \geq 3$, we need the following lemma.

Lemma 2.1. *The map $\{\Omega_n, \cdot\} : \wedge^k(V) \otimes \wedge^m(W) \rightarrow \wedge^k(V) \otimes \wedge^m(W)$ with $k, m \geq 0$ is a vector space isomorphism if $k \neq m$ and $\{\Omega_n, \wedge^k(V) \otimes \wedge^k(W)\} = \{0\}$.*

Proof. We have $\{\Omega_n, \alpha_{i_1} \wedge \dots \wedge \alpha_{i_k}\} = k\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k}$, $\{\Omega_n, \beta_{i_1} \wedge \dots \wedge \beta_{i_m}\} = -m\alpha_{i_1} \wedge \dots \wedge \beta_{i_m}$ and $\{\Omega_n, \alpha_{i_1} \wedge \dots \wedge \alpha_{i_k} \wedge \beta_{j_1} \wedge \dots \wedge \beta_{j_m}\} = (k-m)\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k} \wedge \beta_{j_1} \wedge \dots \wedge \beta_{j_m}$ then the result follows. \square

By a straightforward computation on super Poisson brackets we have the following corollary.

Corollary 2.2. *The restrictions of the differential ∂ from $\alpha \wedge \wedge^i(V) \otimes \wedge^j(W)$ onto $\Omega_n \wedge \wedge^i(V) \otimes \wedge^j(W) \oplus \alpha \wedge \beta \wedge \wedge^i(V) \otimes \wedge^j(W)$ and from $\wedge^i(V) \otimes \wedge^j(W)$ onto $\beta \wedge \wedge^i(V) \otimes \wedge^j(W)$ with $i, j \geq 0$, $i \neq j$ are vector space isomorphisms.*

Let us now give the cases for which $\ker(\partial)$ can be obtained. The following lemma is easy:

Lemma 2.3. *We have $\partial(\wedge^i(V) \otimes \wedge^i(W)) = \partial(\beta \wedge \wedge^i(V) \otimes \wedge^j(W)) = \{0\}$ with $i, j \geq 0$. Moreover, $\partial(\alpha \wedge \beta \wedge \wedge^i(V) \otimes \wedge^j(W)) \subset \partial(\wedge^{i+1}(V) \otimes \wedge^{j+1}(W))$ for all $i, j \geq 0$, $i \neq j$ and*

- (i) $\partial(\alpha \wedge \beta \wedge \wedge^i(V) \otimes \wedge^i(W)) = \beta \wedge \Omega_n \wedge \wedge^i(V) \otimes \wedge^i(W)$,
- (ii) $\partial(\alpha \wedge \wedge^i(V) \otimes \wedge^i(W)) = \Omega_n \wedge \wedge^i(V) \otimes \wedge^i(W)$.

By the reason shown in (i) and (ii) of Lemma 2.3 we set the map

$$\phi_{k_1, k_2, n} : \wedge^{k_1}(\alpha_1, \dots, \alpha_n) \otimes \wedge^{k_2}(\beta_1, \dots, \beta_n) \rightarrow \wedge^{k_1+1}(\alpha_1, \dots, \alpha_n) \otimes \wedge^{k_2+1}(\beta_1, \dots, \beta_n)$$

defined by $\phi_{k_1, k_2, n}(\omega) = \Omega_n \wedge \omega$ then we have the following result.

Proposition 2.4.

(i) *If k is even then*

$$\dim \ker(\partial_k) = \binom{n}{\frac{k}{2}} \binom{n}{\frac{k}{2}} + \sum_{i=0}^{k-1} \binom{n}{i} \binom{n+1}{k-1-i} + \dim \ker \phi_{\frac{k-2}{2}, \frac{k-2}{2}, n} - \binom{n}{\frac{k-2}{2}} \binom{n}{\frac{k-2}{2}}.$$

(ii) *If k is odd then*

$$\dim \ker(\partial_k) = \dim \ker \phi_{\frac{k-1}{2}, \frac{k-1}{2}, n} + \sum_{i=0}^{k-1} \binom{n}{i} \binom{n+1}{k-1-i}.$$

Using the formula $b_k(\mathfrak{g}_{2n+2}) = \dim \ker(\partial_k) + \dim \ker(\partial_{k-1}) - \binom{2n+2}{k-1}$, the binomial identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

and the formula

$$\sum_{i=0}^k \binom{n}{i} \binom{n}{k-i} = \binom{2n}{k}$$

we obtain the following corollary.

Corollary 2.5. *The k^{th} Betti numbers of \mathfrak{g}_{2n+2} are given as follows:*

(i) *If k is even then*

$$b_k(\mathfrak{g}_{2n+2}) = \binom{n}{\frac{k}{2}} \binom{n}{\frac{k}{2}} + 2 \dim \ker \phi_{\frac{k-2}{2}, \frac{k-2}{2}, n} - \binom{n}{\frac{k-2}{2}} \binom{n}{\frac{k-2}{2}}.$$

(ii) If k is odd then

$$b_k(\mathfrak{g}_{2n+2}) = \binom{n}{\frac{k-1}{2}} \binom{n}{\frac{k-1}{2}} + \dim \ker \phi_{\frac{k-1}{2}, \frac{k-1}{2}, n} + \dim \ker \phi_{\frac{k-3}{2}, \frac{k-3}{2}, n} - \binom{n}{\frac{k-3}{2}} \binom{n}{\frac{k-3}{2}}.$$

Hence, it remains to compute $\dim \ker (\phi_{k,k,n})$. Consider the power $\phi_{k_1,k_2,n}^m$ of the map $\phi_{k_1,k_2,n}$ and let

$$K(m, k_1, k_2, n) = \dim \ker (\phi_{k_1,k_2,n}^m)$$

then one has:

Lemma 2.6.

(i) The map

$$\begin{aligned} \theta_{k_1,k_2,n+1}^m : \ker (\phi_{k_1-1,k_2-1,n}^{m+1}) \oplus \ker (\phi_{k_1-1,k_2,n}^m) \oplus \ker (\phi_{k_1,k_2-1,n}^m) \\ \oplus \ker (\phi_{k_1,k_2,n}^{m-1}) \rightarrow \ker (\phi_{k_1,k_2,n+1}^m) \end{aligned}$$

defined by

$$\begin{aligned} \theta_{k_1,k_2,n+1}^m(\omega_1, \omega_2, \omega_3, \omega_4) = \alpha_{n+1} \wedge \beta_{n+1} \wedge \omega_1 + \alpha_{n+1} \wedge \omega_2 + \beta_{n+1} \wedge \omega_3 \\ + \omega_4 - \frac{1}{m} \phi_{k_1-1,k_2-1,n}(\omega_1) \end{aligned}$$

is a vector space isomorphism.

(ii) $K(m, k_1, k_2, n) = K(m+1, k_1-1, k_2-1, n-1) + K(m, k_1-1, k_2, n-1) + K(m, k_1, k_2-1, n-1) + K(m-1, k_1, k_2, n-1)$.

Proof.

(i) The map $\theta_{k_1,k_2,n+1}^m$ is clearly injective. To prove $\theta_{k_1,k_2,n+1}^m$ surjective, let us consider $\omega \in \wedge^{k_1}(\alpha_1, \dots, \alpha_{n+1}) \otimes \wedge^{k_2}(\beta_1, \dots, \beta_{n+1})$ such that $\Omega_{n+1}^m \wedge \omega = 0$. Observe that ω can be written in the form $\omega = \alpha_{n+1} \wedge \beta_{n+1} \wedge \omega_1 + \alpha_{n+1} \wedge \omega_2 + \beta_{n+1} \wedge \omega_3 + \omega_4$ where $\omega_1 \in \wedge^{k_1-1}(\alpha_1, \dots, \alpha_n) \otimes \wedge^{k_2-1}(\beta_1, \dots, \beta_n)$, $\omega_2 \in \wedge^{k_1-1}(\alpha_1, \dots, \alpha_n) \otimes \wedge^{k_2}(\beta_1, \dots, \beta_n)$, $\omega_3 \in \wedge^{k_1}(\alpha_1, \dots, \alpha_n) \otimes \wedge^{k_2-1}(\beta_1, \dots, \beta_n)$ and $\omega_4 \in \wedge^{k_1}(\alpha_1, \dots, \alpha_n) \otimes \wedge^{k_2}(\beta_1, \dots, \beta_n)$. By $\Omega_{n+1}^m \wedge \omega = 0$, we obtain $\Omega_n^m \wedge \omega_2 = \Omega_n^m \wedge \omega_3 = \Omega_n^m \wedge \omega_4 = 0$, $\Omega_n^m \wedge \omega_1 = -m\Omega_n^{m-1} \wedge \omega_4$. It implies $\Omega_n^{m+1} \wedge \omega_1 = 0$ and then $\omega_1 \in \ker (\phi_{k_1-1,k_2-1,n}^{m+1})$. Moreover, $\Omega_n \wedge \omega_1 + m\omega_4 \in \ker (\phi_{k_1,k_2,n}^{m-1})$ means

$$\theta_{k_1,k_2,n+1}^m \left(\omega_1, \omega_2, \omega_3, \omega_4 + \frac{1}{m} \phi_{k_1-1,k_2-1,n}(\omega_1) \right) = \omega.$$

(ii) The assertion (2) follows (1). □

To calculate $K(m, k_1, k_2, n)$, we use the following boundary conditions from the definition of $\phi_{k_1,k_2,n}^m$ in which we assume $\phi_{k_1,k_2,n}^0$ is the identity map:

- (1) $K(0, k_1, k_2, n) = 0$ for all $k_1, k_2, n \geq 0$.
- (2) $K(m, 0, 0, n) = \begin{cases} 0, & \text{if } m \leq n, \\ 1, & \text{if } m > n. \end{cases}$
- (3) $K(m, 0, 1, n) = K(m, 1, 0, n) = \begin{cases} 0, & \text{if } m = 0 \text{ or } n > m, \\ n, & \text{if } 1 \leq n \leq m. \end{cases}$

$$(4) \quad K(m, k_1, k_2, 0) = \begin{cases} 1, & \text{if } m \geq 1, k_1 = k_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

By the condition (2) we extend $K(m, k_1, k_2, n) = 0$ for negative k_1 or k_2 and by the condition (1) we set the condition (5) by $K(-m, k_1, k_2, n) = -K(m, k_1 - m, k_2 - m, n)$.

Lemma 2.7.

$$K(m, k, k, n) = \sum_{p=0}^n \sum_{q=0}^n \binom{n}{p} \binom{n}{q} K(m+n-p-q, k-n+p, k-n+q, 0).$$

Proof. By induction on l , we prove that

$$K(m, k, k, n) = \sum_{p=0}^l \sum_{q=0}^l \binom{l}{p} \binom{l}{q} K(m+l-p-q, k-l+p, k-l+q, n-l).$$

Let $l = n$ to get the lemma. \square

The Betti numbers of \mathfrak{g}_{2n+2} is in the case $m = 1$. By the conditions (4) and (5) we reduce the following.

Corollary 2.8.

$$K(1, k, k, n) = \begin{cases} 0, & \text{if } k < \frac{1}{2}n, \\ \binom{n}{k} \binom{n}{k} - \binom{n}{k+1} \binom{n}{k+1}, & \text{if } k \geq \frac{1}{2}n. \end{cases}$$

Finally, by applying this formula we obtain the Betti number of \mathfrak{g}_{2n+2} according to Corollary 2.5.

3. APPENDIX 1: ANOTHER WAY TO GET THE BETTI NUMBERS OF \mathfrak{g}_{2n+2}

In this part, we shall give another way to get the Betti numbers of \mathfrak{g}_{2n+2} . It is based on the following result.

Proposition 3.1. [6]

Let \mathfrak{g} be an extension of the one-dimensional Lie algebra $\langle z \rangle$ by the Heisenberg Lie algebra \mathfrak{h}_{2n+1} , for some n ,

$$1 \longrightarrow \mathfrak{h}_{2n+1} \longrightarrow \mathfrak{g} \longrightarrow \langle z \rangle \longrightarrow 0$$

such that \mathfrak{g} acts trivially on the center $\mathfrak{z} = \langle w \rangle$ of \mathfrak{h}_{2n+1} . Let $\mathfrak{f} = \mathfrak{g}/\mathfrak{z}$. Then

$$b_k(\mathfrak{g}) = \begin{cases} b_k(\mathfrak{f}) & \text{for } k = 0 \text{ or } k = 1, \\ b_k(\mathfrak{f}) - b_{k-2}(\mathfrak{f}) & \text{for } 2 \leq k \leq n, \\ 2[b_{n+1}(\mathfrak{f}) - b_{n-1}(\mathfrak{f})] & \text{for } k = n+1, \\ b_{k-1}(\mathfrak{f}) - b_{k+1}(\mathfrak{f}) & \text{for } n+2 \leq k \leq 2n, \\ b_{k-1}(\mathfrak{f}) & \text{for } k = 2n+1 \text{ or } k = 2n+2. \end{cases}$$

It is easy to see that \mathfrak{g}_{2n+2} is an extension of the one-dimensional Lie algebra $\langle Y_0 \rangle$ by \mathfrak{h}_{2n+1} . To calculate the Betti numbers of \mathfrak{g}_{2n+2} it needs to find the Betti numbers of the $2n+1$ -dimensional Lie algebra \mathfrak{f} with a basis $\{y, x_1, \dots, x_n, y_1, \dots, y_n\}$ and the Lie bracket

$$[y, x_i] = x_i, \quad [y, y_i] = -y_i$$

for all $1 \leq i \leq n$.

Let $\{y^*, x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*\}$ be the dual basis of $\{y, x_1, \dots, x_n, y_1, \dots, y_n\}$.

Proposition 3.2.(1) *One has*

$$\partial_k \left(y^* \wedge \left(\bigwedge^{k-1} (x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*) \right) \right) = 0.$$

(2) *Assume $j + l = k$ then we have*• *if $j = l$ then*

$$\partial_k \left(\bigwedge^j (x_1^*, \dots, x_n^*) \otimes \bigwedge^l (y_1^*, \dots, y_n^*) \right) = 0,$$

• *if $j \neq l$ then*

$$\partial_k \left(\bigwedge^j (x_1^*, \dots, x_n^*) \otimes \bigwedge^l (y_1^*, \dots, y_n^*) \right) = y^* \wedge \left(\bigwedge^j (x_1^*, \dots, x_n^*) \otimes \bigwedge^l (y_1^*, \dots, y_n^*) \right).$$

Proof. The assertion (1) is obvious. For (2), we use the following computation:

$$\partial_k \left(x_{i_1}^* \wedge \dots \wedge x_{i_j}^* \wedge y_{r_1}^* \wedge \dots \wedge y_{r_l}^* \right) = (j - k) y^* \wedge x_{i_1}^* \wedge \dots \wedge x_{i_j}^* \wedge y_{r_1}^* \wedge \dots \wedge y_{r_l}^*$$

for all $1 \leq i_1 < \dots < i_j \leq n$ and $1 \leq r_1 < \dots < r_l \leq n$. \square

It results the following corollary.

Corollary 3.3. *The Betti numbers of \mathfrak{f} is given as follows:*

$$b_k(\mathfrak{f}) = \binom{n}{\lfloor \frac{k}{2} \rfloor} \binom{n}{\lfloor \frac{k}{2} \rfloor}$$

where $\lfloor x \rfloor$ denotes the integer part of x .

Applying this corollary, we have

$$b_k(\mathfrak{g}_{2n+2}) = \begin{cases} 1 & \text{for } k = 0 \text{ or } k = 1, \\ \binom{n}{\lfloor \frac{k}{2} \rfloor} \binom{n}{\lfloor \frac{k}{2} \rfloor} - \binom{n}{\lfloor \frac{k-2}{2} \rfloor} \binom{n}{\lfloor \frac{k-2}{2} \rfloor} & \text{for } 2 \leq k \leq n, \\ 2 \binom{n}{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{\lfloor \frac{n+1}{2} \rfloor} - 2 \binom{n}{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{\lfloor \frac{n-1}{2} \rfloor} & \text{for } k = n + 1, \\ \binom{n}{\lfloor \frac{k-1}{2} \rfloor} \binom{n}{\lfloor \frac{k-1}{2} \rfloor} - \binom{n}{\lfloor \frac{k+1}{2} \rfloor} \binom{n}{\lfloor \frac{k+1}{2} \rfloor} & \text{for } n + 2 \leq k \leq 2n, \\ 1 & \text{for } k = 2n + 1 \text{ or } k = 2n + 2. \end{cases}$$

and then Theorem 2 is obtained.

4. APPENDIX 2: THE SECOND COHOMOLOGY GROUP OF A FAMILY OF NILPOTENT LIE ALGEBRAS

In this appendix, in the progress of our work, we give the second cohomology of a family of nilpotent Lie algebras that are double extensions of an Abelian Lie algebra (see [3] for more details about these Lie algebras).

Let us denote \mathfrak{g}_{4n+2} a 2-nilpotent quadratic Lie algebra of dimension $4n + 2$ spanned by $\{X, X_1, \dots, X_{2n}, Y, Y_1, \dots, Y_{2n}\}$ where the Lie bracket is defined by $[Y, Y_{2i-1}] = X_{2i}$, $[Y, Y_{2i}] = -X_{2i-1}$, $[Y_{2i-1}, Y_{2i}] = X$ and the bilinear form is given by $B(X, Y) = B(X_i, Y_i) = 1$, zero otherwise. Let $\{\alpha, \alpha_i, \beta, \beta_i\}$ be the dual basis of $\{X, X_i, Y, Y_i\}$. We can check that the associated 3-form I of \mathfrak{g}_{4n+2} is $I = \beta \wedge \Omega$ where $\Omega = \beta_1 \wedge \beta_2 + \beta_3 \wedge \beta_4 + \dots + \beta_{2n-1} \wedge \beta_{2n}$. Therefore, it is easy to see that $\iota_{\mathfrak{g}_{4n+2}}(I) = \text{span}\{\Omega, \beta \wedge \beta_i\}$ for all $1 \leq i \leq 2n$. We have the following proposition.

Proposition 4.1. $\dim(H^2(\mathfrak{g}_{4n+2}, \mathbb{C})) = 8$ if $n = 1$ and $\dim(H^2(\mathfrak{g}_{4n+2}, \mathbb{C})) = 5n^2 + n$ if $n > 1$.

Proof. First we need describe $\ker(\partial_2)$. Let V be the space spanned by $\{\beta, \beta_1, \dots, \beta_{2n}\}$ then $\{I, \omega\} = 0$ for all $\omega \in V \wedge V$. By a straightforward computation, we have

- (1) $\{I, \beta \wedge \alpha_i\} = \{I, \alpha_{2i-1} \wedge \beta_{2i}\} = \{I, \alpha_{2i} \wedge \beta_{2i-1}\} = 0,$
- (2) $\{I, \alpha \wedge \beta\} = I,$
- (3) $\{I, \alpha \wedge \beta_{2i-1}\} = \beta_{2i-1} \wedge \Omega, \{I, \alpha \wedge \beta_{2i}\} = \beta_{2i} \wedge \Omega,$
- (4) $\{I, \alpha \wedge \alpha_{2i-1}\} = \alpha_{2i-1} \wedge \Omega + \beta \wedge \beta_{2i} \wedge \alpha, \{I, \alpha \wedge \alpha_{2i}\} = \alpha_{2i} \wedge \Omega - \beta \wedge \beta_{2i-1} \wedge \alpha,$
- (5) $\{I, \alpha_{2i-1} \wedge \alpha_{2j}\} = -\beta \wedge \beta_{2i} \wedge \alpha_{2j} - \beta \wedge \beta_{2j-1} \wedge \alpha_{2i-1}, \{I, \alpha_{2i} \wedge \alpha_{2j}\} = \beta \wedge \beta_{2i-1} \wedge \alpha_{2j} - \beta \wedge \beta_{2j-1} \wedge \alpha_{2i},$
- (6) $\{I, \alpha_{2i-1} \wedge \beta_{2j}\} = -\{I, \alpha_{2j-1} \wedge \beta_{2i}\} = -\beta \wedge \beta_{2i} \wedge \beta_{2j}, i \neq j,$
- (7) $\{I, \alpha_{2i-1} \wedge \beta_{2j-1}\} = \{I, \alpha_{2j} \wedge \beta_{2i}\} = -\beta \wedge \beta_{2i} \wedge \beta_{2j-1},$
- (8) $\{I, \alpha_{2i} \wedge \beta_{2j-1}\} = -\{I, \alpha_{2j} \wedge \beta_{2i-1}\} = \beta \wedge \beta_{2i-1} \wedge \beta_{2j-1}, i \neq j.$

As a consequence, if $n = 1$ then it is direct that

$$\ker(\partial_2) = V \wedge V \oplus \text{span}\{\beta \wedge \alpha_1, \beta \wedge \alpha_2, \alpha \wedge \beta - \alpha_1 \wedge \beta_1, \alpha_1 \wedge \beta_2, \alpha_1 \wedge \beta_1 - \alpha_2 \wedge \beta_2, \alpha_2 \wedge \beta_1\}.$$

Therefore, we obtain $\dim(H^2(\mathfrak{g}_{4n+2}, \mathbb{C})) = 8$.

In the case $n > 1$ then Ω is indecomposable. Hence,

$$\ker(\partial_2) = V \wedge V \oplus \text{span}\{\beta \wedge \alpha_{2i-1}, \beta \wedge \alpha_{2i}, \alpha \wedge \beta - \sum_{i=1}^n \alpha_{2i-1} \wedge \beta_{2i-1},$$

$$\alpha_{2i-1} \wedge \beta_{2j} + \alpha_{2j-1} \wedge \beta_{2i}, \alpha_{2i-1} \wedge \beta_{2j-1} - \alpha_{2j} \wedge \beta_{2i}, \alpha_{2i} \wedge \beta_{2j-1} + \alpha_{2j} \wedge \beta_{2i-1}\}$$

with $1 \leq i, j \leq n$ and it is easy to check that $\dim(H^2(\mathfrak{g}_{4n+2}, \mathbb{C})) = 5n^2 + n$. \square

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DEPARTMENT OF PHYSICS, HO CHI MINH CITY UNIVERSITY OF PEDAGOGY, 280 AN DUONG VUONG, HO CHI MINH CITY, VIETNAM.

E-mail address: thanhdmi@hcmup.edu.vn